MEMORANDUM
RM-5996-PR
MAY 1969

ANOTHER TYPE OF RISK AVERSION

Emmett Keeler and Richard Zeckhauser

PREPARED FOR:

UNITED STATES AIR FORCE PROJECT RAND

The RAND Corporation
SANTA MONICA • CALIFORNIA

21

MEMORANDUM

RM-5996-PR

MAY 1969

ANOTHER TYPE OF RISK AVERSION

Emmett Keeler and Richard Zeckhauser

This research is supported by the United States Air Force under Project RAND—Contract No. F44620-67-C-0045 monitored by the Directorate of Operational Requirements and Development Plans, Deputy Chief of Staff, Research and Development, Hq USAF. Views or conclusions contained in this study should not be interpreted as representing the official opinion or policy of the United States Air Force.

The RAND Corporation

1700 MAIN ST • SANTA MONICA • CALIFORNIA • 90406

Published by The RAND Corporation

PREFACE

This Memorandum gives a property that utility functions reflecting a certain type of risk-averse behavior must have. The choice of a utility function, giving the value of possible outcomes, is an important step in many decision analyses. For example, these results on risk aversion could be profitably applied in such areas as research allocation over time and the insurance aspects of defense spending as a whole. The Memorandum continues Project RAND's program of research into the methodology of systems analysis.

Richard Zeckhauser, Assistant Professor of Economics, Harvard University, is a consultant to The RAND Corporation.

SUMMARY

Let y be the insurance premium one is just willing to pay to insure against a possible loss λ given an initial wealth position W. It is shown that if

$$s(\lambda, W) = \lambda \left[\frac{-u''(W+\lambda)}{u'(W+\lambda)} \right]$$

is an increasing function of λ, then y/λ is an increasing function of λ. If one is willing to pay a higher persentage to insure against larger possible losses, then one should choose utility functions with $s(\lambda, W)$ increasing in the relevant range.

Examples of functions with this property, and operations on functions that preserve this property are given. The connections with Pratt's risk aversion are presented. It is shown that the premium paid to avoid many-outcome lotteries is an increasing percentage of the scale of the lottery if $s(\lambda, W)$ is increasing for all W.

ACKNOWLEDGEMENTS

We would like to thank Professors John Pratt and Howard Raiffa for helpful comments and suggestions.

CONTENTS

ANOTHER TYPE OF RISK AVERSION

1. INTRODUCTION

Let $u(w)$ be a utility function for wealth, w. Pratt [1] has shown that $r(w) = -u''(w)/u'(w)$ can be interpreted as a measure of local risk aversion at a particular wealth position. He shows that if r is decreasing with w, the risk premium (expected monetary value minus cash equivalent) for a particular lottery will be a decreasing function of pre-lottery wealth. Pratt makes the normative observation that many decisionmakers would feel they ought to pay less for insurance against a given risk the greater their assets. Such a decisionmaker will want to choose a utility function for which $r(w)$ is decreasing.

In contemplating their willingness to pay insurance premiums, some decisionmakers might find it easier to define their preferences in a somewhat different context. Consider an individual faced with what we shall call a simple lottery, a lottery with but two payoffs, one of which is zero. This lottery will result in a loss λ (gain γ) with probability p, but no change in wealth with probability $1 - p$. Given his current wealth, how would the insurance premium (certainty equivalent) he would pay (accept) to avoid the lottery vary with the size of λ?

2. SIZE-OF-RISK AVERSION

We would expect that many individuals would be willing to pay a larger premium as a percentage of the fair actuarial value of an unfavorable simple lottery the greater is the magnitude of the loss included in the lottery. For example, if such an individual would be willing to pay \$130 to insure against a .1 probability of a loss of \$1,000, he would pay more than \$260 to insure against a \$2,000 loss with the same probability. In some sense, then, these individuals are more averse to risk the greater is the size of the potential loss.

Similarly, they may also feel that the certainty equivalent of favorable simple lotteries is a proportionally decreasing function of the gain γ. Such behavior will be called size-of-risk aversion.

More precisely, let W be the initial wealth position. Let w represent total possible assets, so that only positive values need be considered. Let

$$(1) \qquad u(W - y) = p \cdot u(W - \lambda) + (1 - p) \cdot u(W)$$

define y, the insurance premium one is just willing to pay to insure against a loss of size λ that is incurred with probability, p, $0 < p < 1$. A utility function is size-of-risk averse for losses if y/λ is an increasing function of λ, for all $\lambda > 0$.

Similarly, let

$$pu(W + \gamma) + (1 - p)u(W) = u(W + y^*), \tag{2}$$

define y^*, the certainty equivalent one is just willing to accept in place of a chance p to gain γ. A utility function is size-of-risk averse for gains if y^*/γ is a decreasing function of γ.

Theorem 1. *Let* $s(x, W) = x\left[\frac{-u''(W+x)}{u'(W+x)}\right]$

(a) *If* $s(x, W)$ *is increasing for* $m \le x \le 0$, *then* $u(w)$ *is size-of-risk averse for losses less than* $-m$.

(b) *If* $s(x, W)$ *is increasing for* $0 \le x \le M$, *then* $u(w)$ *is size-of-risk averse for gains less than* M.

Proof: Set

$$f(x) = u(W + x) - u(W) \tag{3}$$

Then $f'(x) = u'(W + x)$; $f''(x) = u''(W + x)$.

After substitution, (1) becomes

$$f(-y) = p \cdot f(-\lambda) \tag{4}$$

By differentiating y/λ with respect to λ, we find

(5) y/λ is an increasing [nondecreasing, decreasing] function of $\lambda > 0$ if and only if $y' > [\le, <]\ y/\lambda$, for all λ.

We differentiate the log of both sides of (4) to get

$$\frac{f'(-y)y'}{f(-y)} = \frac{f'(-\lambda)}{f(-\lambda)}. \tag{6}$$

From (5) and (6), we must show

$$\frac{f'(-y)y}{f(-y)} > \frac{f'(-\lambda)\lambda}{f(-\lambda)}. \tag{7}$$

Consider λ fixed. Then y is a function of p. Inequality (7) holds whenever

$$N(p) = f'(-y)yf(-\lambda) - f'(-\lambda)\lambda f(-y) > 0. \tag{8}$$

Clearly, $N(0) = N(1) = 0$. If $N(p)$ is concave, then it must be positive for $0 < p < 1$ and part (A) of the Theorem will be proved.

$$N'(p) = f(-\lambda)[-f''(-y)y'y + f'(-y)y'] + f'(-y)y'f'(-\lambda)\lambda. \tag{9}$$

But differentiating (1) with respect to p, we get $-u'(W - y)y' = u(W - \lambda) - u(W)$, so

$$y' = -\frac{f(-\lambda)}{f'(-y)}. \tag{10}$$

Thus from (9) and (10),

$$N'(p) = [f(-\lambda)]^2\left[\frac{f''(-y)y}{f'(-y)} - 1\right] - f(-\lambda)f'(-\lambda)\lambda.$$

By assumption, $\frac{f''(-y)y}{f'(-y)} = s(-y, W)$ is increasing with $-y$. As p increases, $-y$ decreases, so $N'(p)$ is decreasing. Thus $N(p)$ is concave. Part (B) can be proved similarly.

3. EXAMPLES OF UTILITY FUNCTIONS WITH SIZE-OF-RISK AVERSION

Many simple, familiar, utility functions such as $u(w) = \log(w)$, $u(w) = -e^{-w}$, and $u(w) = w^q$, $0 < q < 1$, all have $u'(w) > 0$, $u''(w) < 0$ and $s(x;W)$ increasing for all positive values.

Changing the scale on which money is measured should not affect decisions. The following theorem proves that linear operations on the argument of a size-of-risk averse utility function preserves size-of-risk aversion.*

<u>Theorem 2</u>. <u>Let</u> $a > 0$; $u_1(x) = u(ax+b)$ <u>is size-of-risk averse for</u> $x_o \leq W \leq x_1$ <u>if and only if</u> $u(x)$ <u>is size-of-risk averse for</u> $ax_o + b \leq W\ ax_1 + b$.

<u>Proof</u>.

$$s_1(x;W) = x\left[\frac{-u_1''(W+x)}{u_1'(W+x)}\right] = x\left[\frac{-u''(a(W+x) + b)\cdot a}{u'(a(W+x) + b)}\right] =$$

$$ax\left[\frac{-u''((W+b) + ax)}{u'((W+b) + ax)}\right] = s(ax,\ W + b).$$

This theorem is the equivalent of Pratt's Theorem 3, which he uses to help him find examples of functions that display risk aversion.

In general, composites and sums of size-of-risk averse functions are not size-of-risk averse. Consider the

---

*It is even simpler to show that $s(x, W)$, like Pratt's measures of risk aversion, are not affected by linear transformations of the utility function.

frequently employed utility function $u(w) = -a^{-bw} - c^{-dw}$ with a, b, c, d > 0 and b > d. It can be demonstrated that s(x; W) is increasing for $0 \leq W + x \leq W + \frac{b^2}{(b - d)^2}$. In addition, it is increasing for all positive W + x if

(11) $$\frac{b - d}{b} W + \frac{b}{b - d} \geq \log \frac{ab(b - d)}{cd^2} .$$

4. RELATIONSHIP BETWEEN DIFFERENT MEASURES OF RISK AVERSION

Since $s(x, W) = xr(W + x)$, there evidently is a relationship between risk aversion and size-of-risk aversion. There is a more direct relationship between size-of-risk aversion and $r^*(w) = wr(w)$, which Pratt shows to be a measure of proportional risk aversion. Proportional risk aversion relates risk premiums and gambles when both are measured as proportions of wealth.

Let I be the interval over which we wish to examine the relationship between these measures, and let D be the set of values of W for which $s(x; W)$ is increasing for all $W + x$ in I. The table shows the relationships that can be inferred.

Part A

If D contains	then for $W + x$ in I, we have
$W = 0$	increasing proportional risk aversion.
W approaching $+\infty$	nonincreasing risk aversion.

Part B

If risk aversion for all w in I is	then for $w = W + x$ in I
nonincreasing	$s(x; W)$ increases for $w \leq W$.
constant	$s(x; W)$ increases for all w.
nondecreasing	$s(x; W)$ increases for $w \geq W$.

For simplicity, use Theorem 2 to make $W = 1$ and all of I positive. Then

$$(12) \quad s(x,1) = (1+x)r(1+x) - r(1+x) = r^*(1+x) - r(1+r).$$

Thus an individual has size-of-risk aversion if the difference between proportional and absolute risk aversion (measured in units of initial wealth) is increasing.

It can be shown that for very small losses and gains any utility function that shows risk aversion and that has u''' continuous or locally bounded will display size-of-risk aversion.

5. RISK AVERSION WITH RESPECT TO SCALE OF LOTTERY

At times an individual may be faced with a lottery that gives him a positive probability of a number of different outcomes. It may be of interest to know how the amount that he will pay to avoid or accept to give up the lottery will vary with a scale parameter that multiplies by all the possible payoffs.

If the risk premium $\pi(W, x\tilde{z})$ as a proportion of x is to increase with x, for an arbitrary fair gamble $\tilde{z}$, it suffices to have $s(x; W)$ increasing for all W in $[0, +\infty)$. This will be the case for example, if $u(w) = -e^{-kw}$. By reference to the table, this is a somewhat stronger condition than the hypotheses of the theorem below. Let $E(\tilde{z}) = 0$ and suppose π is defined by

$$(13) \qquad u(W - \pi(W, x\tilde{z})) = E(u(W + x\tilde{z})).$$

Theorem 3. $\dfrac{\pi(W, x\tilde{z})}{x}$ is an increasing [decreasing] function of x if w

$$w\frac{u''(w)}{u'(w)} \text{ and } \frac{-u''(w)}{u'(w)}$$

are non-increasing [nondecreasing] functions of w.

Proof. Normalize $\tilde{z}$ by multiplying by W/x. Then by Theorem 6 of Pratt, since $z\dfrac{-u''(z)}{u'(z)}$ is nondecreasing

$$(14) \qquad \frac{\pi(W, W\tilde{z})}{W} \leq \frac{\pi(W + \epsilon, (W + \epsilon)\tilde{z})}{W + \epsilon}.$$

By Theorem 2 of Pratt, since $\frac{-u''(z)}{u'(z)}$ is nonincreasing

$$(15) \qquad \frac{\pi(W+\epsilon,\ (W+\epsilon)\tilde{z})}{W+\epsilon} \leq \frac{\pi(W,\ (W+\epsilon)\tilde{z})}{W+\epsilon}$$

If $\frac{-u''(w)}{u'(w)}$ is nonincreasing, then for $w > 0$, $w\,\frac{u''(w)}{u'(w)}$ is decreasing. If $w\,\frac{u''(w)}{u'(w)}$ is nonincreasing then for $w < 0$, $\frac{-u''(w)}{u'(w)}$ is decreasing. Thus the inequality in either (14) or (15) must be strict. We combine (14) and (15) to complete the proof.

This result can be extended to nonneutral gambles. Suppose $E(\tilde{z}) = \mu$. Then $E(u(W + x\tilde{z})) = E(u(W + x\mu + x(\tilde{z} - \mu)))$, so

$$(16) \qquad u(W - \pi(W, x\tilde{z})) = u(W + x\mu - \pi_1\ (W + x\mu,\ x(\tilde{z} - \mu)))$$

Since $E(\tilde{z} - \mu) = 0$, $\frac{\pi_1}{x}$ is increasing, so $\pi(W, x\tilde{z}) = \frac{-x\mu + \pi_1}{x}$ is increasing.

6. CHOICE OF THE UTILITY FUNCTION

If individuals wish to make choices that reflect size-of-risk aversion, this may influence the choice of the utility function they use to represent their preferences. There will be no problem with utility functions that have constant risk aversion. For decreasing risk aversion to be satisfactory in this regard it suffices that they have nondecreasing proportional risk aversion. A counter-example shows that a weaker condition is not sufficient.

The utility function $u(w) = -e^{1/w}$ shows decreasing proportional risk aversion as well as decreasing risk aversion. Measuring w in units of \$100,000, if initial wealth is \$10,000 there will not be size-of-risk aversion for simple lotteries offering gains of more than \$15,000.

There are many relationships and complementarities between the two concepts, risk aversion and size-of-risk aversion. If the former concept is a bit more general and tractable, the latter may be better understood on an intuitive basis. From a normative standpoint, we believe that consideration of size-of-risk aversion as well as of risk aversion should be part of the process through which a utility function is chosen.

REFERENCE

Pratt, John W., "Risk Aversion in the Small and in the Large," Econometrica, Vol. 32, Jan.-Apr. 1964, pp. 122-136.

DOCUMENT CONTROL DATA

1. ORIGINATING ACTIVITY THE RAND CORPORATION		2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP
3. REPORT TITLE ANOTHER TYPE OF RISK AVERSION		
4. AUTHOR(S) (Last name, first name, initial) Keeler, Emmett and Richard Zeckhauser		
5. REPORT DATE May 1969	6a. TOTAL No. OF PAGES 22	6b. No. OF REFS. 1
7. CONTRACT OR GRANT No. F44620-67-C-0045	8. ORIGINATOR'S REPORT No. RM-5996-PR	
9a. AVAILABILITY / LIMITATION NOTICES DDC-1	9b. SPONSORING AGENCY United States Air Force Project RAND	

10. ABSTRACT

A formulation incorporating the concept of "size-of-risk" aversion into the process of selecting a utility function. This concept extends and complements Pratt's normative observation of risk aversion, namely, that as wealth increases, many decisionmakers would feel that they ought to pay less insurance against a given risk. However, as the size of potential loss increases, decisionmakers are more averse to risk and would be willing to pay a larger premium. They display what is known as (positive) size-of-risk aversion. In selecting a utility function, both concepts should be considered.

11. KEY WORDS

Decisionmaking
Numerical methods and processes
Mathematics
Probability
Operations research
Econometrics